Int. J. Solids Structures Vol. 27, No. 10, pp. 1287–1293, 1991
Printed in Great Britain.

0020–7683/91 $3.00 + .00

POSTBUCKLING OF AN EXTERNALLY PRESSURIZED RING WITH A HINGE

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(*Received* 1 *November* 1989; *in revised form* 5 *May* 1990)

Abstract—An inextensible, elastic circular ring is weakened by a hinge. Due to external pressure, the ring buckles and collapses in three stages. The large deformation equations are formulated and integrated numerically. The introduction of a hinge lowers the buckling pressure by more than half. Postbuckling configurations show marked asymmetry.

1. INTRODUCTION

Rings are often used as stiffeners for thin shell structures such as submarines. If the ring is weakened by a hinge, the critical buckling pressure will be reduced. An equivalent hinge may be created by loosened fasteners or internal cracks due to defect or damage. On the other hand, the problem of a hinged ring is similar to externally pressurized tubes with a longitudinal seam caused by defect or incomplete welding. The only difference in the analyses is that the flexural rigidity is EI for the ring and E thickness^3/12(1-Poisson ratio^2) for the tube.

If the ring does not have a hinge, the critical buckling pressure for the elastic perfect ring is $3EI/R^3$, where R is the radius of the underformed ring (Lévy, 1884; Timoshenko and Gere, 1964). For the ring with one hinge, Timoshenko and Gere (1964) erroneously concluded that the buckling pressure is zero by taking a limit on the base-pinned arch. The correct buckling pressure of $1.3923EI/R^3$ was found by Wang (1985) using a perturbation on the circle. The present paper studies the postbuckling, large deformation properties of a collapsing perfectly elastic ring weakened by a hinge. We assume the ring is thin enough such that it can be considered as an inextensible elastica.

2. FORMULATION

Figure 1 shows a ring of original radius R being compressed by external pressure p'. Let the origin of Cartesian axes (x', y') be situated diametrically opposite the hinge where x' is along the axis of symmetry. A local moment balance yields

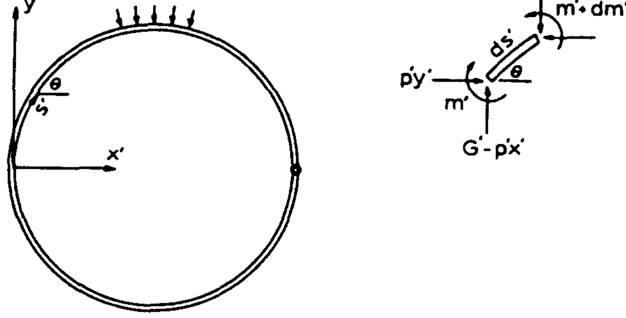

Fig. 1. The coordinate system. Small circle is the hinge.

$$dm' = (G' - p'x')\,ds'\cos\theta - p'y'\,ds'\sin\theta \tag{1}$$

where G' is the still unknown vertical force at the origin, s' is the arc length and θ is the local angle of inclination. The local moment m' is related to curvature difference by

$$m' = EI\left(\frac{d\theta}{ds'} + \frac{1}{R}\right). \tag{2}$$

The Cartesian coodinates are related by

$$\frac{dx'}{ds'} = \cos\theta, \quad \frac{dy'}{ds'} = \sin\theta. \tag{3}$$

We normalize all lengths by R, the forces by EI/R^2, the pressure by EI/R^3 and drop the primes. Equations (1)–(3) become

$$\frac{d^2\theta}{ds^2} = (G - px)\cos\theta - py\sin\theta \tag{4}$$

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta. \tag{5}$$

The determination of the buckling pressure is briefly described as follows. Perturb about the circle by setting $\theta = (\pi/2) - s + \varphi$ where $|\varphi| \ll 1$. Differentiate eqn (4) twice and use eqn (5) to yield the linearized equation

$$\varphi'''' + (p+1)\varphi'' = 0. \tag{6}$$

The solution of φ is odd in s

$$\varphi = c_1 s + c_2 \sin(\sqrt{p+1}\,s). \tag{7}$$

Since the moment is zero at $s = \pi$,

$$\varphi'(\pi) = 0 = c_1 + c_2\sqrt{p+1}\cos(\sqrt{p+1}\,\pi). \tag{8}$$

On the other hand, the fact that $y(0) = y(\pi) = 0$ gives

$$\int_0^\pi \varphi \sin s\, ds = 0 = c_1\pi - \frac{c_2}{p}\sin(\sqrt{p+1}\,\pi). \tag{9}$$

For nontrivial c_1, c_2 eqns (8) and (9) yield the equation

$$\pi p\sqrt{p+1} + \tan(\sqrt{p+1}\,\pi) = 0. \tag{10}$$

The lowest eigenvalue is the buckling pressure $p_b = 1.3923$.

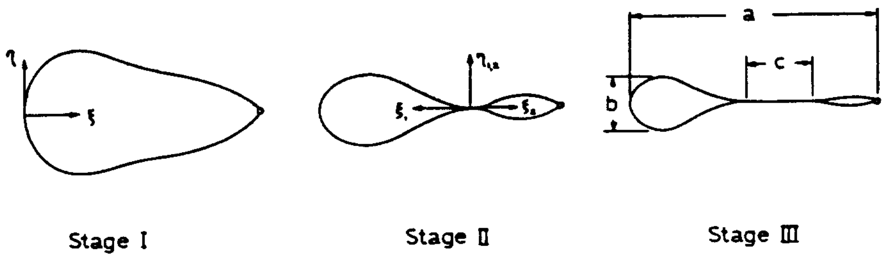

Fig. 2. The three buckling stages.

After initial buckling, we expect the ring to collapse in three stages (Fig. 2). For Stage I the pressure is greater than the buckling pressure $p_b = 1.3923$ and less than p_t, at which the sides of the ring just begin to touch. Stage II is when there is only one point in contact, $p_t < p < p_s$ where p_s is the minimum pressure for a segment to be in contact. Stage III is when $p_s < p$ and the ring becomes two separate loops with one flattened middle section.

3. NUMERICAL INTEGRATION

For Stage I the boundary conditions at the origin are

$$\theta(0) = \frac{\pi}{2}, \quad x(0) = y(0) = 0 \tag{11}$$

and at the hinge

$$y(\pi) = 0, \quad \frac{d\theta}{ds}(\pi) = -1. \tag{12}$$

Equations (4), (5), (11) and (12) are more easily integrated by first using renormalization

$$s = p^{-1/3}t, \quad x = p^{-1/3}\xi, \quad y = p^{-1/3}\eta, \quad G = p^{2/3}g. \tag{13}$$

The governing equations become

$$\frac{d^2\theta}{dt^2} = (g-\xi)\cos\theta - \eta\sin\theta \tag{14}$$

$$\frac{d\xi}{dt} = \cos\theta, \quad \frac{d\eta}{dt} = \sin\theta. \tag{15}$$

For given $g > 0$ we guess $d\theta/dt(0) < 0$ and integrate eqns (11), (14) and (15) as an initial value problem by the fifth order Runge–Kutta–Fehlberg algorithm. A step size of 0.05 is found to be sufficient. The integration terminates when η becomes zero again, say at $t = t^*$. The value of $d\theta/dt(0)$ is adjusted such that the following conditions are satisfied

$$\eta(t^*) = 0, \quad t^*\frac{d\theta}{dt}(t^*) + \pi = 0. \tag{16}$$

Then the pressure is obtained *a posteriori*:

$$p = \left(\frac{t^*}{\pi}\right)^3. \tag{17}$$

The original variables can be recovered from eqn (13). Our numerical results show $p_t = 2.2609$ at touching.

Stage II is more difficult. Let the origin be situated at the touching point and separate the ring into two sections as shown in Fig. 2. The initial conditions for the looped section without the hinge are

$$\eta_1(0) = \xi_1(0) = \theta_1(0) = 0. \tag{18}$$

For given $\mathrm{d}\theta_1/\mathrm{d}t(0)$ eqns (14) and (15) are integrated until η_1 becomes zero at $t = t_1^*$. The constant g_1 is adjusted such that $\theta_1(t_1^*) = -\pi/2$. For the hinged section the initial conditions are

$$\eta_2(0) = \xi_2(0) = \theta_2(0) = 0, \quad \frac{\mathrm{d}\theta_2}{\mathrm{d}t}(0) = \frac{\mathrm{d}\theta_1}{\mathrm{d}t}(0). \tag{19}$$

The constant g_2 is adjusted such that at $\eta_2(t_2^*) = 0$, we need

$$s_1(t_1^*) + s_2(t_2^*) = \pi \tag{20}$$

and

$$\frac{\mathrm{d}\theta}{\mathrm{d}s}(s_2(t_1^*)) = -1. \tag{21}$$

Equations (20) and (21) combine to give

$$p = \left[-\frac{\mathrm{d}\theta_2}{\mathrm{d}t}(t_2^*)\right]^{-3} \tag{22}$$

$$\pi\left[-\frac{\mathrm{d}\theta_2}{\mathrm{d}t}(t_2^*)\right]^{-1} - t_2^* = t_1^*. \tag{23}$$

Stage II ends when $\mathrm{d}\theta/\mathrm{d}t(0) = 0$ and two sections become independent. We find the corresponding pressure is $p_s = 4.0517$.

Stage III has three sections. The looped section now becomes similar. Using the initial conditions $\theta_1(0) = \mathrm{d}\theta_1/\mathrm{d}t(0) = 0$ the solution is

$$G_1 = p^{2/3} g_1 = 0.83770 p^{2/3} \tag{24}$$

$$s_1^* = p^{-1/3} t_1^* = 3.42169 p^{-1/3} \tag{25}$$

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}s}(s_1(t_1^*)) = p^{1/3}\frac{\mathrm{d}\theta_1}{\mathrm{d}t}(t_1^*) = -2.10544 p^{1/3}. \tag{26}$$

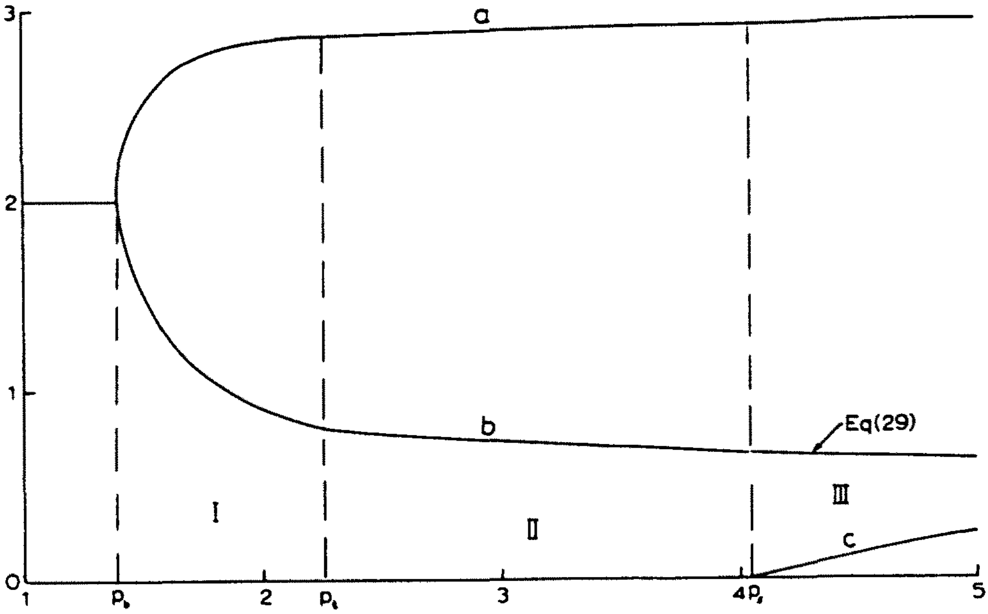

Fig. 3. Width a, height b, and contact length c versus pressure.

The middle section is flat, say with an unknown length of t_3^*. The hinged section has the same initial conditions $\theta_2(0) = d\theta_2/dt(0) = 0$ but there is no similarity. Pick any g_2 and integrate till $\eta_2(t_2^*) = 0$. Then

$$p = \left[-\frac{d\theta_2}{dt}(t_2^*) \right]^{-3} \tag{27}$$

$$t_3^* = \pi p^{1/3} - t_2^* - 3.42169 \geq 0. \tag{28}$$

4. RESULTS AND DISCUSSION

Let a be the maximum width, b be the maximum height and c be the length of the section in contact. Figure 3 shows the variation of these geometric parameters as pressure is increased. For $p < p_b = 1.3923$ the ring is circular and both width and height have a value of 2. In Stage I postbuckling, height rapidly decreases while width rapidly increases. The changes are much slower in Stages II and III. The length of contact c starts to increase from zero in Stage III. Since the looped section in Stage III has the maximum height, one can use similarity to deduce, for $p > p_s = 4.0517$,

$$b = 2p^{-1/3}\eta_{1\,\max} = 1.0638p^{-1/3}. \tag{29}$$

Let m and M be the minimum and maximum moment experienced, normalized by EI/R. From eqn (2) the moment is equal to curvature plus one. The minimum or most negative moment always occurs at the tip of the looped end at $s = 0$. The most positive moment occurs, surprisingly, not at the point of contact for Stages II and III but somewhere between the tip and the contact point. Figure 4 shows that the absolute magnitudes of m and M become larger as pressure is increased from p_b. The largest changes occur in Stage I. There exists a discontinuity in slope between Stages I and II. From similarity in Stage III we deduce, for $p > p_s$,

$$M = 0.3480p^{1/3} + 1 \tag{30}$$

$$m = -2.1054p^{1/3} + 1. \tag{31}$$

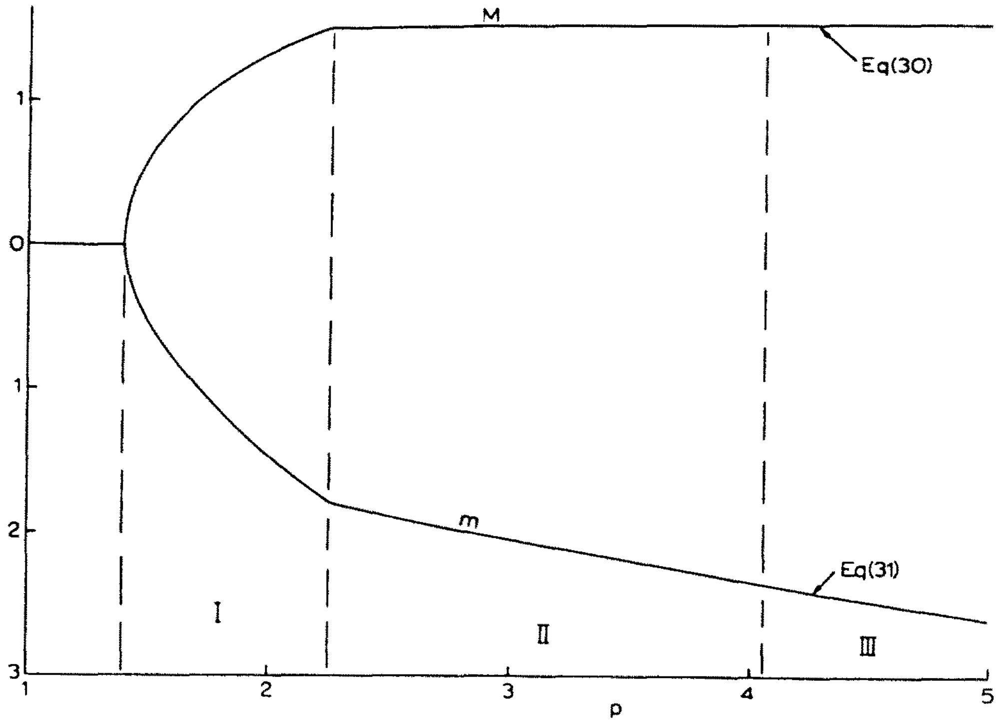

Fig. 4. The maximum moment M and the minimum moment m.

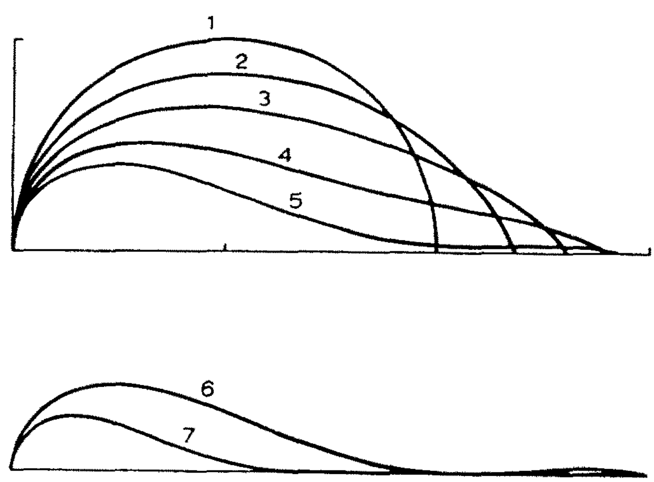

Fig. 5. Postbuckling configurations. 1, $p < 1.3923$; 2, $p = 1.456$; 3, $p = 1.578$; 4, $p = 1.838$; 5, $p = 2.204$; 6, $p = 2.2609$; 7, $p = 10$.

Note that in all Stages $|m| > |M|$.

Figure 5 shows the buckled large deformation configurations for various pressures. There is no lateral symmetry.

Due to asymmetry, the present problem is much more difficult than the symmetric hingeless ring. We have devised some numerical schemes to simplify the integrations into, at most, a one parameter shooting algorithm.

The postbuckling of a ring without hinges was studied by Flaherty *et al.* (1972). They found $p_b = 3$, $p_t = 5.247$ and $p_s = 10.34$. For a ring weakened by a hinge, we find the corresponding pressures are lowered by more than half: $p_b = 1.3923$, $p_t = 2.2609$ and $p_s = 4.0517$. Such significant decreases in strength should be taken into account in any design of pressured vessels.

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